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Crossover Scaling Functions for 2*d* Vesicles, and the Yang–Lee Edge Singularity

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An account is given of recent progress in establishing an exact formula for the critical scaling function of self-avoiding loops in two dimensions, weighted by their area.

KEY WORDS: Critical behavior; self-avoiding walks.

INTRODUCTION

This talk is about some recent exact results on crossover scaling functions in two dimensions. Although much of the work of Michael Fisher has rightly emphasised the importance of explaining complicated physics in terms of simple theories, he has also, from early on, derived many elegant results for exactly solvable models.

The work I will be discussing is contained in two recent papers by Richard, Guttmann, and Jensen[1] and myself,[2] although some of the ideas relate to older work of Parisi and Sourlas,[3] which has very recently been put on a much more systematic footing by Brydges and Imbrie[4] (see Imbrie's contribution to this meeting.) However, it really all goes back to a seminal paper of Leibler, Singh, and Michael Fisher,[5] who were concerned with the statistical mechanics of vesicles, biological closed membranes. Although much of this paper was devoted to the effects of local curvature-dependent terms in the energy on the overall shape of the vesicle, these authors also considered, through Monte Carlo simulations and scaling

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analysis, a very simplified model for two-dimensional vesicles, in which the wall corresponds to a self-avoiding random loop. The ensemble of all such loops of a fixed perimeter is weighted by the pressure difference p between the inside and outside, which couples to the internal area of the loop (the $2d$ analog of the volume.)

The generating function for this ensemble is

$$Z = \sum_{\text{rooted loops}} x^L e^{-pA}$$

where x is the monomer fugacity. The sum over *rooted* loops, i.e., those which pass through a given point, is to eliminate overcounting. When $p = 0$, Z is known to have a critical point at $x = x_c$, at the which the mean loop size diverges. Leibler, Singh, and Fisher[5] argued that for $p > 0$ there is a *crossover phenomenon*, from self-avoiding loops to branched polymers, as illustrated in Fig. 1, and described by a *crossover scaling function*[6]

$$\langle A \rangle = -\frac{\partial}{\partial p} \ln Z = \langle L \rangle^{2\nu}\, \mathrm{Y}(p \langle L \rangle^{2\nu})$$

or equivalently

$$Z_{\text{sing}} = p^{\theta} F((x_c - x)\, p^{-\phi})$$

This scaling function must satisfy a number of constraints:

- when $p = 0, N(L) \sim L^{\alpha-2}(x_c^{-1})^L$, so that $Z \sim (x_c - x)^{1-\alpha}$ and $F(u) \sim u^{\theta/\phi}$ with $\theta/\phi = 1 - \alpha$;
- we expect that $\langle A \rangle \sim \langle L \rangle^{2\nu}$, which implies $\phi = 1/2\nu$, where $1/\nu$ is the fractal dimension of the loop.

The new result[1,2] is an exact form for the scaling function:

$$F(u) = \frac{\mathrm{Ai}'(u)}{\mathrm{Ai}(u)}$$

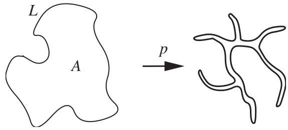

Fig. 1. The crossover from self-avoiding loops to branched polymers as the pressure difference p is increased.

where

$$\mathrm{Ai}(u) \propto \int_{-\infty}^{\infty} e^{iut+it^3/3}\,dt$$

is the Airy function.

Moreover, it may be argued[2] that $\phi = \frac{2}{3}$ and $\theta = \frac{1}{3}$, which implies the previously conjectured values $\alpha_{\mathrm{SAL}} = \frac{1}{2}$, $\nu_{\mathrm{SAL}} = \frac{3}{4}$.

q-ALGEBRAIC APPROACH

Richard, Guttmann, and Jensen[1] pointed out that simpler solvable problems (staircase polygons, convex polygons, etc.) have generating functions $Z(x, q \equiv e^{-p})$ which satisfy q-algebraic equations. For example, *staircase polygons* (see Fig. 2), which consist of a pair of non-intersecting random walks which may only move to the right or upwards on a square lattice[2], have a generating function satisfying

$$Z(x, q) = \frac{qx^2}{1-qx} + \frac{x+Z(x, q)}{1-qx}\,Z(qx, q)$$

These authors *assume* that $Z_{\mathrm{SAP}}(x, q)$ satisfies some equation

$$\sum_n \sum_{k_1, \ldots, k_n} a_{k_1, \ldots, k_n} \prod_{j=1}^{n} Z(q^{k_j} x, q) = b(x, q)$$

with a scaling solution $Z \sim (1-q)^{\theta} \times F((x_c - x)(1-q)^{-\phi})$ and *also* that $\theta = \frac{1}{3}$, $\phi = \frac{2}{3}$. An asymptotic scaling analysis then shows that F satisfies a Riccati equation

$$F'(u) = aF(u)^2 - bu$$

with solution $F(u) \propto (d/du) \ln \mathrm{Ai}(u)$.

Although the central assumption here might seem rather strong, there is no doubt that the result is correct. The formula predicts specific values for the universal ratios of the moments $\langle A^n \rangle / \langle A \rangle^n$, which are related to derivatives of Z with respect to p, evaluated at $p = 0$. Richard *et al.*[1] enumerated self-avoiding loops on the lattice (polygons) up to very large values of the perimeter, and were able to verify the predictions for these ratios to a spectacular degree of accuracy.

[2] These are nothing but the simplest case of the so-called vicious walker problem introduced by M. E. Fisher in 1984.[7]

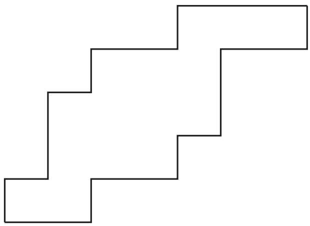

Fig. 2. A staircase polygon, consisting of two mutually avoiding directed random walks.

FIELD-THEORETIC APPROACH

The emergence of the Airy function may, however, be understood from a completely different argument, in which the crossover from self-avoiding loops to branched polymers is analyzed from each end, using methods of continuum field theory.

Self-Avoiding Loops

These may be described by the $n \to 0$ limit of a theory of a (complex) $O(n)$ field $\vec{\phi}$, in which each (oriented) loop carries a factor n. If we imagine a unit current around each loop flowing around each loop in the sense of its orientation, the corresponding current density is represented in the continuum by the U(1) current $J_\mu \sim (1/2i)(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*)$

The area of a given loop may be written as

$$A = \int |x_1 - x_2| \, \delta(y_1 - y_2) \, dy_1 \, dy_2$$

$$= \iint G_{\lambda\sigma}(r_1 - r_2) \, J_\lambda(r_1) \, J_\sigma(r_2) \, d^2 r_1 \, d^2 r_2$$

where $G_{\lambda\sigma}$ is the Green function for a U(1) gauge field $\mathscr{A}$. This is a well-known result: in a $2d$ gauge theory, there is a linear potential between opposite charges, so the expectation value of a Wilson loop obeys a strict area law.

Thus we can write the area-weighted partition function in field-theoretic language as

$$Z = \langle e^{-pA} \rangle_{\mathrm{SAL}} = \langle e^{-\sqrt{p} \int J^\lambda \mathscr{A}_\lambda \, d^2 r} \rangle_{\mathrm{SAL}, \mathscr{A}}$$

where the gauge field is integrated over with weight $\exp(-\int F^{\lambda\sigma} F_{\lambda\sigma} \, d^2 r)$.

This is just the $n \to 0$ limit of the Abelian Higgs model ($n = 2$ would be the Landau–Ginzburg model for a superconductor.) However, this case has special features:

- at $n = 0$ there are no vacuum corrections to the gauge field propagator $G_{\lambda\sigma}$ (like the "quenched" approximation in lattice gauge theories)
- $\mathscr{A}$ couples to a conserved current J.

Together, these imply that the gauge coupling p is not renormalized. Its RG equation is

$$dp/d\ell = 2p$$

to all orders, so p flows to a fixed point at ∞, where the irrelevant variable p^{-1} has the RG eigenvalue -2.

Branched Polymers in *d* Dimensions

There are many different microscopic models of branched polymers, all of which seem to fall into the same universality class. From a field theory point of view, following Parisi and Sourlas,[3] it is useful to visualise branched polymers as tree Feynman diagrams, which, as is well known, correspond to the solution of classical field equations. For example

$$(-\nabla^2 + m_0^2)\,\psi = h + u_3\psi^2 + \cdots$$

which would generate the trees, shown in Fig. 3, with each segment corresponding to the Green function of $-\nabla^2 + m_0^2$, each branching vertex carrying factor u_3, and each branch tip a factor h. The solution of this equation may be written using a functional delta-function:

$$\int \mathscr{D}\omega\, \mathscr{D}\psi\; e^{\int \omega(-\nabla^2\psi + V'(\psi))\, d^d r}\ \text{(Jacobian)}$$

where

$$\text{Jacobian} = \int \mathscr{D}\bar{\chi}\, \mathscr{D}\chi\; e^{\int \bar{\chi}(-\nabla^2 + V''(\psi))\,\chi\, d^d r},$$

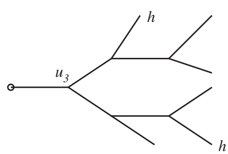

Fig. 3. Tree Feynman diagrams as a model for branched polymers.

an integral over grassman fields. However, this model does not include self-avoidance: it turns out to be sufficient to incorporate repulsion into the end-points of the tree by letting $h \to h + i\,\delta h(r)$ and averaging over $\delta h(r)$ with weight $\exp(-(1/\Delta)\int(\delta h)^2\,d^d r))$. After rescaling ω, the action is of the form

$$S = \frac{1}{\Delta}\int (\omega(-\nabla^2\psi + V'(\psi)) - \omega^2 + \bar{\chi}(-\nabla^2 + V''(\psi))\,\chi)\,d^d r$$

It was the remarkable observation of Parisi and Sourlas[3] that this is supersymmetric. This is seen by supplementing the d commuting euclidean coordinates with anticommuting coordinates $(\theta, \bar{\theta})$, such that $\int d\bar{\theta}\,d\theta = 0$ and $\int d\bar{\theta}\,d\theta\,\bar{\theta}\theta = 1$, and defining the superfield $\Psi(r, \bar{\theta}, \theta) \equiv \psi(r) + \bar{\theta}\chi(r) + \theta\bar{\chi}(r) + \bar{\theta}\theta\omega(r)$ In this notation S may be written

$$S = \frac{1}{\Delta}\int (\Psi(-\nabla^2_{SS})\,\Psi + V(\Psi))\,d^d r\,d\theta\,d\bar{\theta}$$

where $\nabla^2_{SS} = \nabla^2 + 4\partial^2/\partial\theta\,\partial\bar{\theta}$.

This has several important properties:

- it exhibits SUSY under rotations which leave $r^2 + \theta\bar{\theta}$ invariant.

- it exhibits *dimensional reduction*: correlation functions whose arguments are restricted to a $d-2$-dimensional subspace are same as those for a non-susy theory in $d-2$ dimensions, whose action is

$$S' = \frac{1}{\Delta}\int \left(\frac{1}{2}\psi(-\nabla^2)\,\psi + V(\psi)\right) d^{d-2} r$$

The basic mathematical reason for this is the identity $\int d^2 r\,d\theta\,d\bar{\theta}\,f(r^2 + \theta\bar{\theta}) \propto \int d(r^2)\,f'(r^2) = -f(0)]$

- Δ has dimension $(\text{length})^{-2}$ (not affected by loop corrections, otherwise susy would be broken,) so that

- under the RG, $d\Delta/d\ell = 2\Delta$, so that Δ flows to ∞, where Δ^{-1} is irrelevant. (It is a classic example of a *dangerously* irrelevant variable:[8] it cannot be set equal to zero.)

On the basis of comparing the above two formulations of the problem, it is natural to conjecture that

$$\Delta \propto p,$$

so that, for $d = 2$ in particular,

$$Z(x, p)_{\text{sing}} = \langle \Psi(0) \rangle_{\text{susy}} = \langle \psi \rangle_{S'} = (\partial/\partial x) \ln Z_1$$

where

$$Z_1 = \int e^{(x\psi - V(\psi))/p} \, d\psi$$

For this to have required scaling form $p^{\theta} F((x_c - x) p^{-\phi})$, $x\psi - V(\psi)$ must have a simple critical point at $x = x_c$. The simplest choice is $x\psi - V(\psi) = -(x_c - x)\psi + \frac{1}{3}\psi^3$. After rotating the contour and rescaling $\psi \to p^{1/3}\psi$, we find the required scaling form with $\theta = \frac{1}{3}$, $\phi = \frac{2}{3}$ and $F(u) = \mathrm{Ai}'(u)/\mathrm{Ai}(u)$.

COMMENTS AND PUZZLES

- This is probably the first known example of an exact scaling function of two thermodynamic variables for a nontrivial isotropic critical point;
- from this point of view, the Airy function arises as a partition function in $d = 0$, with no apparent connection to a q-algebraic structure (however, it should be possible to see whether expected corrections to scaling are consistent with such a structure);
- Brydges and Imbrie[4] have shown an exact mapping between a particular model of branched polymers in d dimensions and a repulsive gas (at negative fugacity) in $d - 2$ dimensions [using susy!]. Its scaling limit is $i\psi^3$ field theory, also known as the Yang–Lee theory, whose critical properties were first analyzed by M. E. Fisher.[9] That the hard core gas at negative fugacity should be in the same universality class as the Yang–Lee singularity has been studied by Lai and Fisher.[10]
- the above argument can be generalized to higher order critical points $V \propto \psi^{k+2}$: for $p \to 0$ these lead to new candidates for multicritical points of self-avoiding loops, whose physical interpretation as yet obscure (the exponents for $k = 2$ do not correspond to those for the theta-point.)
- however, the scaling dimensions do coincide with Flory values, and with those found by Saleur (1992) on the basis of a presumed twisted $N = 2$ susy for the $d = 2$ CFTs describing these theories.
- these arguments give yet another "derivation" of the result $\nu = \frac{3}{4}$ for self-avoiding walks in two dimensions—it is high time this was made rigorous.

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